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# SOME OLD AND NEW PERSPECTIVES FOR DIFFERENTIAL GAMES

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# 1. ZERO SUM DIFFERENTIAL GAMES

**DYNAMICS:**

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha, \beta) \quad (0 \leq t \leq \tau) \\ \mathbf{x}(0) = x, \end{cases}$$

**CONTROLS:**

$$\begin{cases} \alpha : [0, \tau] \to A = \text{ control for the maximizing player} \\ \beta : [0, \tau] \to B = \text{ control for the minimizing player} \end{cases}$$

**PAYOFF:**

$$J^{\alpha,\beta}(x) = \int_0^\tau r(\mathbf{x}, \alpha, \beta)\, dt + g(\mathbf{x}(\tau)).$$

$r$ is the **running payoff** and $g$ is the **terminal payoff**.

**MINIMAX CONDITION:**

$$\min_{b\in B}\max_{a\in A}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\} = \max_{a\in A}\min_{b\in B}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\}$$

for all $x, p$.

**VALUE FUNCTION:**

$$u(x) = \text{payoff to max player, if each plays optimally}$$

**THEOREM**

*The value function $u$, if continuous, solves the* **Hamilton–Jacobi–Isaacs (HJI)** *equation*

$$H(Du, x) = 0$$

*in the viscosity sense, for the* **game theory Hamiltonian**

$$H(p,x) = -\min_{b\in B}\max_{a\in A}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\}.$$

# References


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• R. Isaacs, *Differential Games. A mathematical theory with applications to warfare and pursuit, control and optimization*, Wiley, 1965.

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# Problems for theory applied to "real" examples

- Hamiltonian $H = H(p, x)$ is nonconvex in $p$.
- Hamiltonian $H$ is not $C^1$ (and therefore problems with characteristics = optimal trajectories).
- The value function $u$ need not be continuous.
- Boundary conditions are unclear and/or hard to interpret in viscosity sense.

Failure of minimax condition is usually not a problem, at least for examples in Isaacs' and Lewin's books.

# AN (UNNAMED) EXAMPLE OF ISAACS

**DYNAMICS:**

$$\begin{cases} \dot{x} = c(y) + \cos\alpha \\ \dot{y} = 2\beta + \sin\alpha, \end{cases}$$

where $0 \leq \alpha \leq 2\pi$ and $-1 \leq \beta \leq 1$.

$$y \mapsto c(y) \text{ is positive, increasing.}$$

The game is played in the upper half plane $U := \mathbb{R}^2 \cap \{y \geq 0\}$.

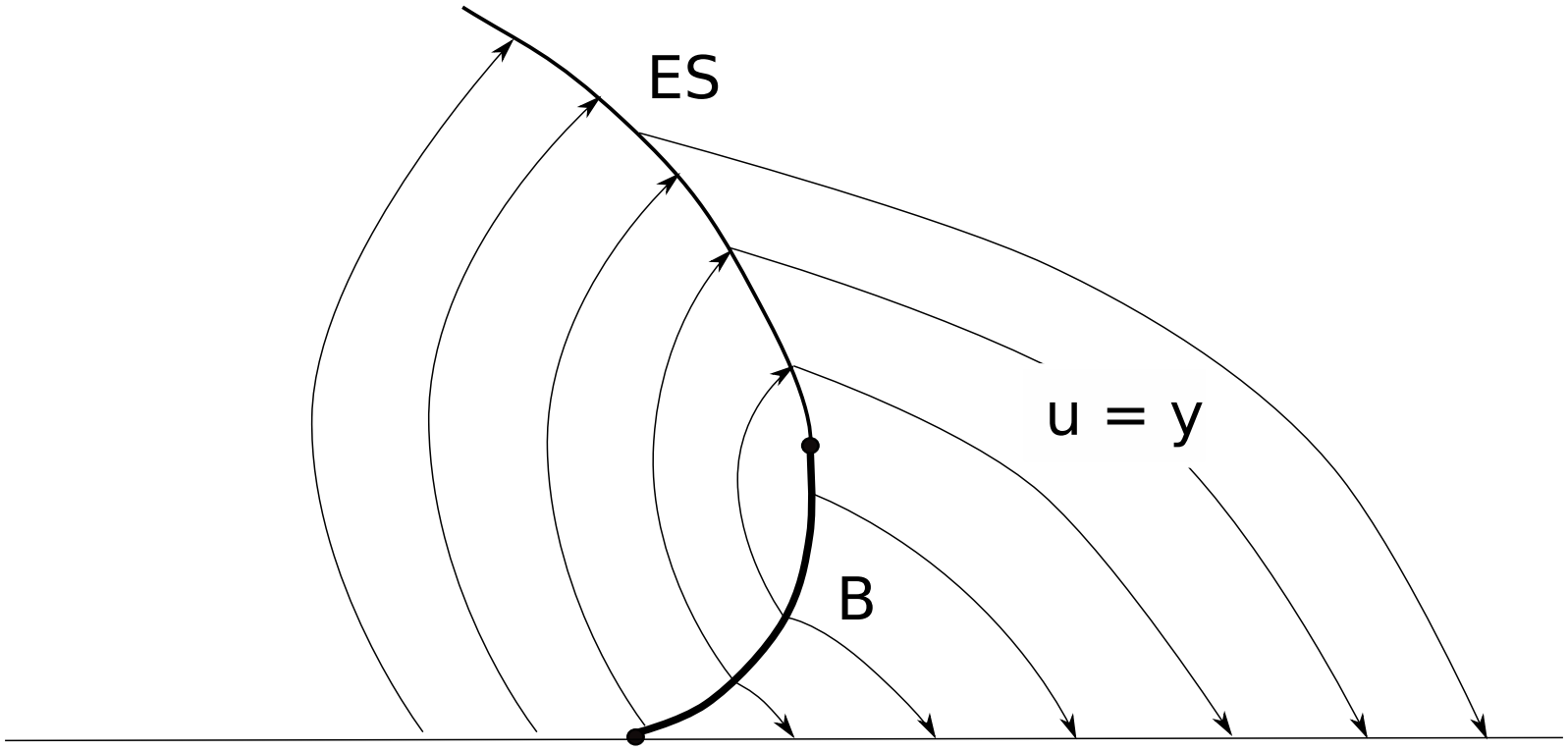
**PAYOFF:**

$$J^{\alpha,\beta} = \tau = \text{ exit time from } U \text{ along the positive } x\text{-axis.}$$

**HJI EQUATION:** The value function $u$ (= time to exit) solves

$$H(Du, y) = -c(y)u_x - |Du| + 2|u_y| - 1 = 0 \quad \text{in } U.$$

Boundary condition is $u = 0$ on $\{x \geq 0, y = 0\}$.



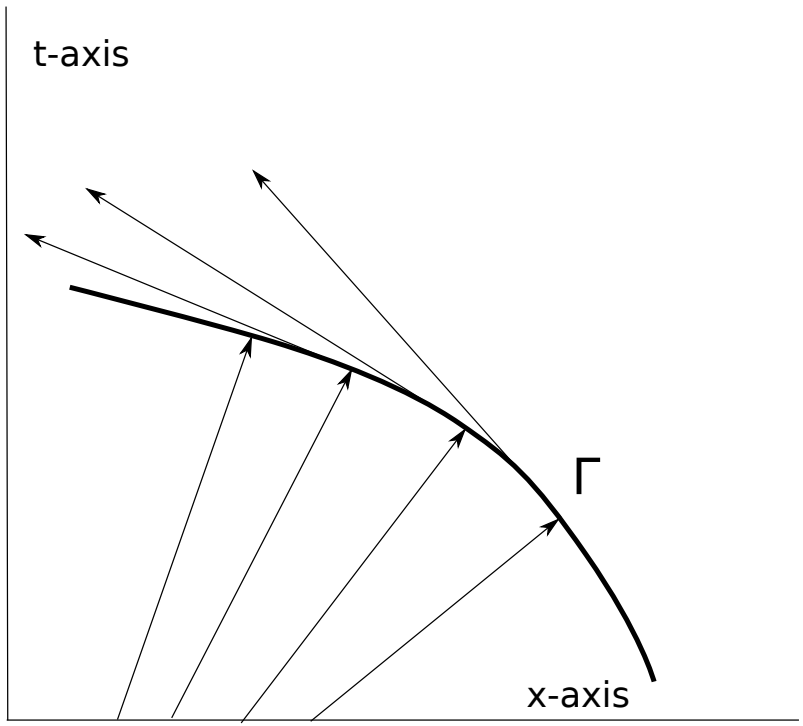

$B =$ **barrier**; $ES =$ "**equivocal surface**"

# 2. NONCONVEX HAMILTON-JACOBI PDE

Study the initial-value problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where $H$ is smooth, but not convex.



The central difficultly for nonconvex Hamilton–Jacobi PDE is that backwards characteristics can hit "shocks", along which the gradient is discontinuous.

New work of F. Rezakhanlou: Backwards characteristics as martingales.

**FORMULAS FOR SOLUTIONS:**

- If $H$ is convex,

$$u(x,t) := \inf_{y \in \mathbb{R}^n} \left\{ g(y) + tL\left(\tfrac{x-y}{t}\right) \right\} \qquad \text{(Hopf–Lax)}$$

where $L = H^*$

- If $g$ is convex and has linear growth,

$$u(x,t) := \sup_{z \in \mathbb{R}^n} \left\{ x \cdot z - g^*(z) - tH(z) \right\} \qquad \text{(Hopf)}$$

**THEOREM** (ENVELOPE REPRESENTATION FORMULA)

*For $\mathcal{L}^{n+1}$-a.e. point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ there exists a Radon probability measure $\gamma_{x,t}$ on $\mathbb{R}^n$ such that*

$$u(x,t) = \int_{\mathbb{R}^n} g(y) + (x-y) \cdot Dg(y) - tH(Dg(y))\, d\gamma_{x,t}.$$

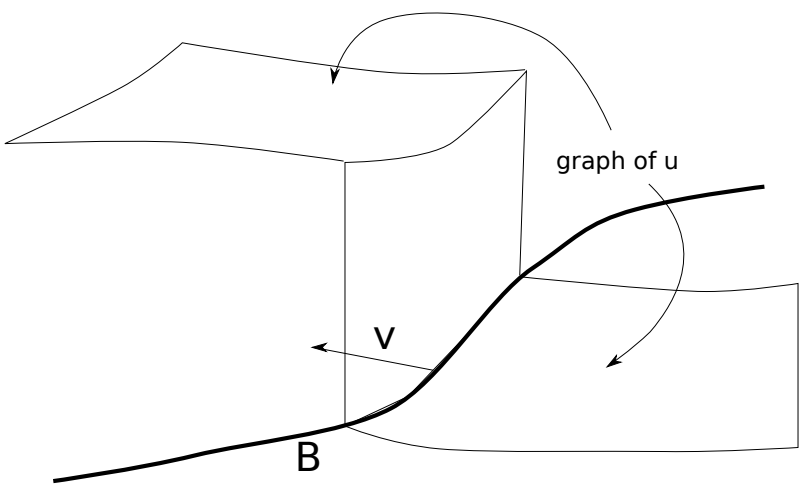
Furthermore,

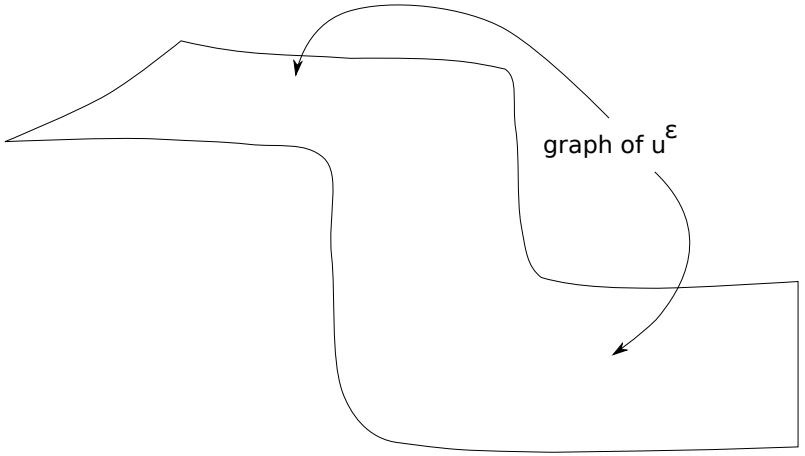
$$u_t(x,t) = -\int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}, \quad Du(x,t) = \int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t},$$

and

$$H\left(\int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t}\right) = \int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}.$$

# BARRIERS (= surfaces of discontinuity)




$$H(Du, x) = 0 \quad \text{in } U.$$

**APPROXIMATION:**

$$-\epsilon \Delta u^\epsilon + H(Du^\epsilon, x) = 0 \quad \text{in } U.$$

**Assume:**

$$u^\epsilon \leq u^+ + o(1), \;\; u^\epsilon \geq u^- - o(1).$$

The **recession function** associated with $H = H(p, x)$ is

$$K(p, x) := \lim_{\lambda \to \infty} \frac{H(q + \lambda p, x)}{\lambda}.$$

**THEOREM**

(i)

$$\begin{cases} H(Du^+ + \lambda\nu, x) \leq 0 \\ H(Du^- + \lambda\nu, x) \geq 0 \end{cases} \quad \textit{for all } \lambda \geq 0.$$

(ii) *In particular,*

$$K(\nu, x) = 0.$$

This is a PDE for the barrier (Isaacs). We can in principle solve for the location of the discontinuities, before we solve the full HJI equation!

# 3. NONZERO SUM GAMES: PRINCIPAL-AGENT PROBLEMS

SANNIKOV'S MODEL (simplified)

**DYNAMICS:**

$$dX = Adt + dB,$$

where $B$ is a one-dimensional Brownian motion.

**CONTROLS:**

$$\begin{cases} C = \text{ principal's control } (= \text{payments to agent}) \\ A = \text{agent's control of dynamics.} \end{cases}$$

**PAYOFFS:**

$$\begin{cases} J^P = \text{principal's payoff} = E\left(\int_0^\infty e^{-t}(A - C)\, dt\right) \\ J^A = \text{agent's payoff} = E\left(\int_0^\infty e^{-t}(g(C) - h(A))\, dt\right) \end{cases}$$

# How can we make the agent do what we want?

In particular, how can the principal encourage the agent to perform the action $A^*$?

KEY POINT #1: The agent's actions and the Brownian motion are unobservable by the principal; she can only observe $X$.

**DYNAMICS REINTERPRETED:**

$$\begin{aligned} dX &= A^* dt + dB \\ &= A dt + dB^A \end{aligned}$$

KEY POINT #2: $B$ is a Brownian motion under the original probability measure $P$ and $B^A$ is a Brownian motion under a new probability measure $P^A$.

**PAYOFFS REINTERPRETED:**

$$\begin{cases} J^P = E^A\left(\int_0^\infty e^{-t}(A - C)\, dt\right) \\ J^A = E^A\left(\int_0^\infty e^{-t}(g(C) - h(A))\, dt\right) \end{cases}$$

**NEW DYNAMICS:** Given $A^* = a(W)$ and $C^* = c(W)$, solve

$$\begin{aligned} dW &= (W - g(C^*) + h(A^*))dt + h'(A^*)dB \\ &= (W - g(C^*) + h(A^*) - A^* h'(A^*))dt + h'(A^*)dX \\ W(0) &= w. \end{aligned}$$

The principal will therefore pay the agent $C^* = c(W)$, depending upon the values of $W$, which she can compute in terms of $X$.

**THEOREM** (Sannikov optimality condition for agent)

*Assume*

$$h \text{ is convex.}$$

*Then it is optimal for the agent to use the control $A^*$.*

# New proof of optimality condition for agent

$$\begin{aligned} d(e^{-t}W) &= e^{-t}\left[(-g(C^*)+h(A^*))dt + h'(A^*)dB\right] \\ &= e^{-t}\left[(-g(C^*)+h(A^*)+h'(A^*)(A-A^*))dt + h'(A^*)dB^A\right] \end{aligned}$$

Integrate from 0 to $\infty$:

$$-w = \int_0^\infty e^{-t}(-g(C^*)+h(A^*)+h'(A^*)(A-A^*))\,dt + \int_0^\infty e^{-t}h'(A^*)\,dB^A$$

Take expected value with respect to $P^A$ and rewrite:

$$w = E^A\left(\int_0^\infty e^{-t}(g(C^*)-h(A))\,dt\right) + E^A\left(\int_0^\infty e^{-t}(h(A)-h(A^*)-h'(A^*)(A-A^*))\,dt\right)$$

By convexity, the last term is nonnegative, and is zero for $A = A^*$. □

**EXERCISE**: Use dynamic programming to rederive this optimality condition. What is the agent's value function?

**THEOREM** (Optimality condition for principal)

*The principal's value function $u = u(w)$ solves the HJB equation*

$$\max_{a,c} \left\{ \frac{(h'(a))^2}{2} u'' + (w - g(c) + h(a))u' - u + a - c \right\} = 0$$

*Select*

$$a(w), c(w)$$

*giving the max in the HJB equation.*

• I.Ekeland, *How to build stable relationships with people who lie and cheat*, www.ceremade.dauphine.fr/∼ekeland/Articles/13-03-08-ArticleMilano.pdf

Many fascinating PDE/optimality issues remain open.